Introduction to Stochastic Processes	**Midterm**	**Max score: 50**
Semester I 2025/26	**Date: 10th September 2025.**	**Time:** 1 hour and 30 min

Instructions: Complete and precise solutions will receive full credit. Imprecise or partial solutions will recieve very little or no credit. In writing down partial solutions try to indicate the gaps as clearly as possible. If you are using a result from class then please state and verify the hypothesis required for conclusion.

1. (**15 points**) Let X_n be a Markov chain on $S = \{0, 1, \dots\}$ with transition matrix $\mathbf{P}$ with

$$p_{ij} = \begin{cases} 1 & \text{if } i = j = 0 \\ p_i & \text{if } i \geq 1, j = i + 1 \\ 1 - p_i & \text{if } i \geq 1, j = i - 1 \\ 0 & \text{otherwise} \end{cases}$$

where we assume $0 < p_i < 1$ for every $i \geq 1$. Let $h : S \to [0, 1]$ to be given by

$$h(i) = \mathbb{P}(T^0 < \infty \mid X_0 = i),$$

with $T^0 = \inf\{n \geq 0 : X_n = 0\}$. Find h.

2. (**15 points**) Let $\{S_n\}_{n\geq 0}$ be a simple symmetric random walk on $\mathbb{Z}$ starting at 0.

   (a) For $a < b, a, b \in \mathbb{Z}$, $n \geq 1$ show that

   $$\mathbb{P}(a \leq S_n \leq b) \leq (b - a + 1)\mathbb{P}(S_n \in \{0, 1\})$$

   (b) $\lim_{n\to\infty} \mathbb{P}(a \leq S_n \leq b) = 0$.

3. (**15 points**) Let $0 < p < 1$, $L \geq 1$ and $S = \{0, 1, 2, \dots, L\}$. Assume $\mathbb{P}(S_0 = 1) = 1$ and S_n be a Markov chain with transition matrix $\mathbf{P} = [p_{ij}]_{i,j\in\mathbb{Z}}$ given by

$$p_{ij} = \begin{cases} p & \text{if } j = i + 1, i \neq L \\ 1 - p & \text{if } j = i - 1, i \neq 0 \\ p & \text{if } j = 0, i = L \\ 1 - p & \text{if } j = L, i = 0 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Sketch the graph induced by this Markov chain on the vertex set S.

   (b) Show that the chain is irreducible and determine its periodicity.

   (c) Find the stationary distribution π of the chain.

4. (**15 points**) Let $\mathbb{T}_2$ be a rooted binary tree, with root ρ. That is, a graph with the vertex set given by

$$\cup_{n=1}^{\infty}\{0, 1\}^n \cup \{\rho\}$$

and edge set given by

$$\{\{\rho, x\} : x \in \{0, 1\}\} \cup \{\{x, p(x)\} : x \in \{0, 1\}^n, n \geq 2\},$$

where for $n \geq 2$, and $x = (x_1, x_2, \dots, x_n)$ we set the parent $p(x) = (x_1, x_2, \dots, x_{n-1})$. On $\mathbb{T}_2$, consider the weight function μ to be given by

$$\mu(\{x, p(x)\}) = \beta^{|x|} \text{ for } x \in \mathbb{T}_2$$

where β is a positive number and $\mid x \mid$ denote the graph distance to the root ρ. Let X_n denote the canonical random walk on $(\mathbb{T}_2, \mu)$. Show that X_n is transient if and only if $\beta > \frac{1}{2}$.